Report on Problem 17: Water Bottle
(GYPT 2018)

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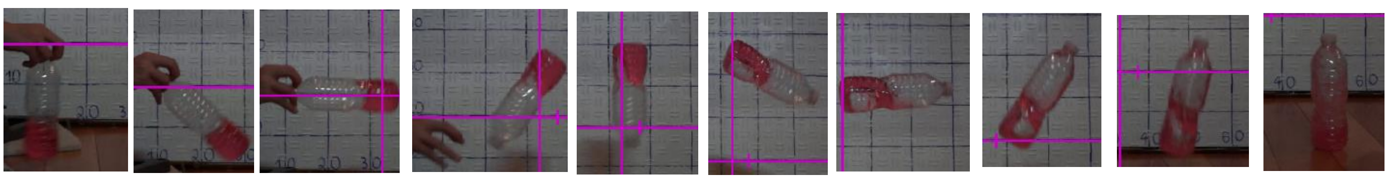
1. INTRODUCTION

In this report I will attempt to look into the following problem:

*"The current craze of water bottle flipping involves launching **a partially filled plastic bottle** into the air so that it performs **a somersault** before landing on a horizontal surface in a **stable, upright position**. **Investigate** the phenomenon and determine the **parameters** that will result in a **successful** flip."*

In other words, one has to determine the parameters of a successful flip of the water bottle by observing its position in the air and the flowing of the water inside the bottle. To do so, I perform a series of experiments and compare them with theoretical explanations.

1.1 Basic Explanation



The water bottle flip is influenced by the water inside the bottle and outside factors, namely the gravitational force and the release height of the water bottle. The movement of the water inside the bottle during the flip triggers the change in the angular velocity and changes the point of the center of mass. For the bottle to land successfully upright, the gravitational force on the center of mass needs to pass through the surface of the bottom of the bottle. At first the center of mass is at the base of the bottle, it then shifts to the middle, and moves towards the bottom as the water flows back down to the bottom. Because of the conservation of the angular momentum, this slows down the angular velocity causing the bottle to spin slower than in the beginning, so that the bottle is influenced by gravity and falls to the ground. This particular movement of the water is only possible when one has the right amount of water. Given a full water bottle or an empty water bottle, the center of mass stays at the same place, so that the bottle's spin does not slow and has no chance to successfully land. This is why it is almost impossible to achieve the perfect water bottle flip in these scenarios.

1.2 Background Research

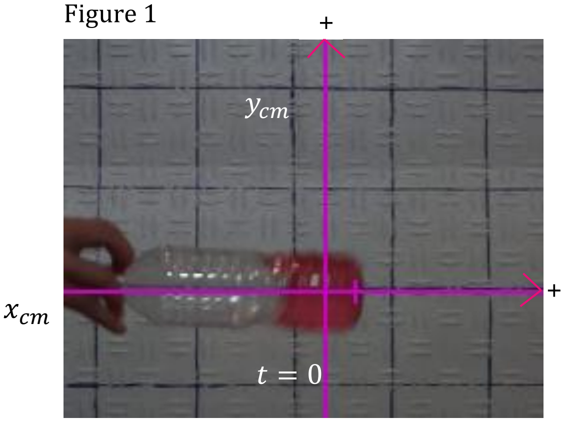
To get a feeling of the water bottle flip, I started flipping water bottles hundreds of times, observed its motion, and figured out strategies for a successful flip. In addition to this, I also watched many videos of experts and even babies who could do the water bottle flip. They typically did not throw the bottle very far or very high. What they all had in common was that the water bottle was never full nor empty, but was filled to about a quarter to one third of the height of the water bottle. I also read an interview of one of the water bottle flip experts, who gave tips on how to get the ideal water bottle flip. The three main advice points were to: a) fill the bottle up to a third, b) throw consistently, and c) impart spin. After understanding the most important parts about the water bottle flip, I proceeded to research the mechanics of the center of mass and how to calculate it. I also investigated minimum requirements for high speed cameras.

2. OWN RESEARCH

In the following chapter, I will explain my research process so far. After many tries of getting as many successful water bottle flips as possible, I combined the theoretical and the experimental approach to determine formulas for the motion of the water bottle. To do that I experimented with suitable setups.

2.1 Theoretical Model

Figure 1



I chose the following coordination system, where the location of the center of mass ($x_{\text{center of mass}}$, $y_{\text{center of mass}}$) and time t are 0 and rotation angle φ is 90^0 when the bottle is roughly horizontal and being released from the hand (Figure *1*).

For all but the smallest fill heights, the mass of the plastic bottle is negligible compared to water mass for fill heights in my experiment (see section 3.1).

As soon as the bottle is released, the only force acting on the center of mass is the gravitational force. Therefore the **motion of center of mass** from $t = 0$ onwards is simply:

$$x_{cm}(t) = v_{x0} \cdot t \tag{1}$$

$$y_{cm}(t) = -\frac{1}{2} g \cdot t^2 + v_{y0} \cdot t \tag{2}$$

v_{x0} and v_{y0} are the initial velocities when the bottle is released at $t = 0$.

The cap and the bottom of the bottle rotate around the center of mass with the same angular velocity. However the radius of this rotation movement changes with time as the center of mass moves along the bottle when the water spreads out. Therefore the **motion of the cap** is:

$$x_{cap}(t) = x_{cm}(t) - (1 - \gamma_{cm}(t)) \cdot l \cdot \cos(\varphi(t)) \tag{3}$$

$$y_{cap}(t) = y_{cm}(t) - (1 - \gamma_{cm}(t)) \cdot l \cdot \sin(\varphi(t)) \tag{4}$$

where l is the length of the bottle, $\varphi(t)$ is the rotation angle, and $\gamma_{cm}(t)$ is the relative distance of the center mass from the bottom of the bottle (Figure *2*).

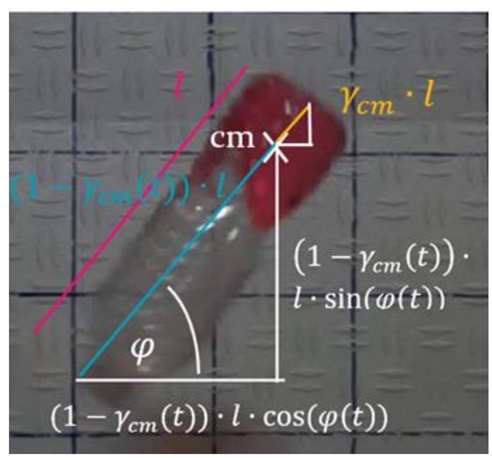


Figure 2

Similarly the **motion of the bottom** of the bottle is:

$$x_{bottom}(t) = x_{cm}(t) + \gamma_{cm}(t) \cdot l \cdot \cos(\varphi(t)) \tag{5}$$

$$y_{bottom}(t) = y_{cm}(t) + \gamma_{cm}(t) \cdot l \cdot \sin(\varphi(t)) \tag{6}$$

How does the rotation angle depend on time t? “Tracker” provided an analysis, which shows that the rotation angle can be modeled with the help of the logistics function with steepness parameter k.

$$\varphi(t) \approx \frac{2\pi}{1+e^{-k(t-t_0)}} - \frac{\pi}{2} = \frac{2\pi}{1+e^{-k\left(t-\frac{\ln(3)}{k}\right)}} - \frac{\pi}{2} \qquad \text{for successful flips} \qquad (7)$$

t_0 is the inflection point of $\varphi(t)$. Putting $\varphi(0) = 0$, therefore $t_0 = \frac{\ln(3)}{k}$. Equation (7) describes the full 360^0 rotation starting at -90^0 until 270^0, hence a successful flip. Therefore, the angular velocity is:

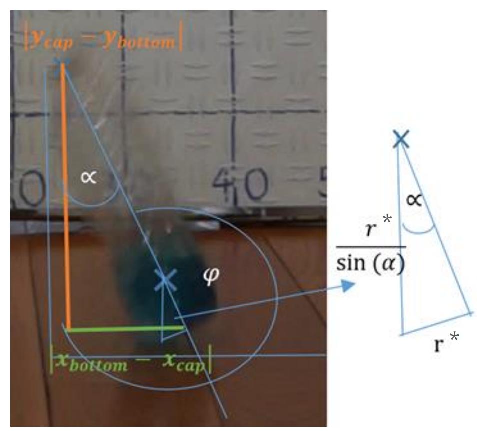
$$\omega(t) = \dot{\varphi}(t) = \frac{2\cdot\pi\cdot k\cdot e^{-k\left(t-\frac{\ln(3)}{k}\right)}}{\left[1+e^{-k\left(t-\frac{\ln(3)}{k}\right)}\right]^2} \qquad \text{for successful flips} \qquad (8)$$

$$\omega(0) = \frac{3\pi k}{8} \qquad \text{for successful flips} \qquad (9)$$

How does $\gamma_{cm}(t)$ change with time? Initially, the distance of the center of mass from the bottom of the bottle $\gamma_{cm}(t) \cdot l$ stays close to half the fill height of the bottle, i.e., $\frac{h}{2}$ because of the rapid rotation of the bottle (Figure 2). This strictly applies only for fill heights that are not too small (see section 3.1). A simple approximation of $\gamma_{cm}(t)$ is a dynamic pulse (Gaussian distribution frequently seen in nature).

$$\gamma_{cm}(t) \approx a \cdot \mathrm{e}^{\frac{-(t-t_{max})^2}{2c^2}} + \frac{h}{2l}$$

(10)Where a (height of pulse) and c (measure of width of pulse) are fit parameters. t_{max} is time at $\varphi(t_{max}) = 180°$ yielding with equation (7).



Under which condition does the bottle stand up right when it hits the ground? Clearly the gravitational force vector from the center of mass needs to pass through the bottom surface of the bottle. Therefore as indicated in the figure xx, and using the equations of motion for cap and bottom (eq. 3, 4, 5, and 6):

$$\tan(\propto) = \frac{|x_{bottom} - x_{cap}|}{|y_{cap} - y_{bottom}|} = \frac{|\cos(\varphi(t))|}{|\sin(\varphi(t))|} = \frac{r*}{\gamma_{cm}(t)\cdot l} \qquad (11)$$

where r is the radius of the bottom surface of the bottle, r* is the radius to where the gravitational force vector from CM passes through the bottle bottom, and h is the fill height of the water bottle. The bottle stands upright, if $r^* \leq r$. In other words, the bottle flip is successful, if at time t_E when the bottle hits the ground:

$$\gamma_{cm}(t_E) \cdot l \, \frac{|\cos(\varphi(t_E))|}{|\sin(\varphi(t_E))|} \leq r \qquad (12)$$

Time t_E is calculated by setting equation 2 to the y value of the ground. H is the release height above ground and $\frac{r}{\sin(\alpha)}$ is the distance from the center of mass to the ground.

$$y_{cm}(t_E) = -\frac{1}{2} g \cdot {t_E}^2 + v_{y0} \cdot t_E = -H + \frac{r}{\sin(\alpha)}$$

This yields: $t_E = \frac{1}{g} \cdot (v_{y0} + \sqrt{{v_{y0}}^2 + 2g(H - \frac{r}{\sin(\alpha)}})$ (13)

Equation 12 together with equations 13, 7, and 10 describe the condition for a successful bottle flip. In other words, success is determined by the following parameters: release height H, bottom radius r, bottle length l, mass of empty bottle, fill height h, initial velocity v_{y0}, and initial angular velocity $\omega(0)$. Because of the fixed geometrical seize of the bottle, the only parameters left to vary for the "Poland Spring" bottle are the release height, fill height, initial velocity, and initial angular velocity.

Equations 12 and 13 can be simplified for successful and almost successful throws. The center of mass has moved down to the lowest point and $\gamma_{cm}(t_E) \cdot l$ can be approximated by the upright standing bottle with ρ being the constant $\frac{m_{water}}{h}$:

$$Center\ of\ mass(h) = \frac{m_{water} \cdot \frac{h}{2} + m_{Bottle} \cdot \frac{l}{2}}{m_{water} + m_{Bottle}} = \frac{h^2 \cdot \rho + m_{Bottle} \cdot l}{h \cdot \rho + m_{Bottle}} \quad (14)$$

In addition to the right angle with which the angle has to touch the ground (12), another condition that increases the chance of the bottle landing upright, is that the angular velocity at the time t_E when the bottle hits the ground is as low as possible. The angular velocity $\omega(t_E)$ can be derived from the conservation of angular momentum:

$$I(t) \cdot \omega(t) = constant \quad (15)$$

While the moment of inertia is fixed for a solid body (e. g., my frozen ice experiment) it changes with time due to the movement of the water in the bottle.

2.2 Experimental Setup and Data Analysis

Figure 3

Figure 4

At first, I built a machine (Figure 3) to exactly reproduce the water bottle flip each time. However, I found out very soon that this would not be possible because every suspension (see Figure 4) that I tried out failed to perform consistently over a wider range of parameters. This is why I decided to simply use my own arm and wrist to throw the bottle. Another reason why I think my experimental setup (Figure 5) is suitable is that in reality, people are supposed to do the water bottle flip and not machines. My experiment is reproducible, as I can fully track initial parameters with a high-speed camera.

In my experimental setup, the release height of the bottle and the fill height of the bottle can be held constant subsequently. The release height does not change because the metal bar restricts my arm from going any higher or lower as my arm needs to bang against it. The fill height can be controlled easily by adding or removing water. This leaves only the variation of the parameters of initial velocity and angular velocity. An additional fixed parameter is that I also used one type of plastic water bottle from the company "Poland Spring". I focused on only this type as it is the standard sized water bottle in the US. Its full height is 20.5 cm, its bottom surface has a diameter of 5.5 cm, a cap surface diameter of 2.5 cm, and an empty mass of 12 grams.

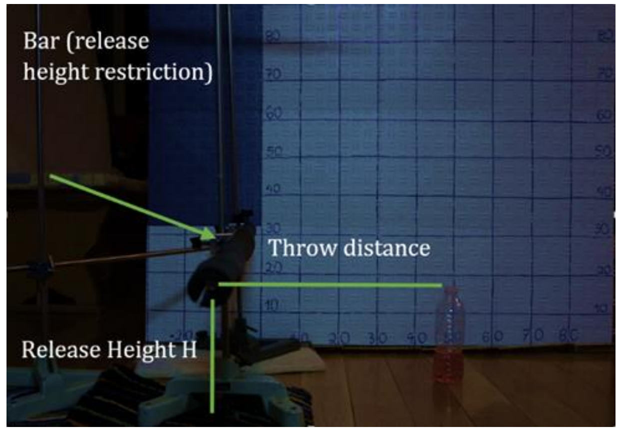


Figure 5

To fully understand the phenomenon, I recorded 477 throws with a high speed camera (Sony Nex-7) at 60 frames per second. I used the program "Tracker" 4.11.0 (physlets.org/tracker) to analyse the selected videos. This allows measuring of distances, velocities and rotations at a reasonable precision ($\pm$2mm; $\pm 17ms$).

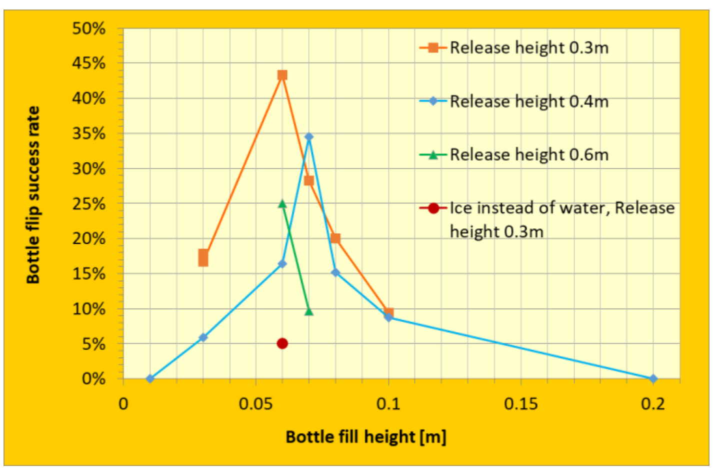


Figure 6

Figure 6 reports the overall success rates for the 477 bottle flips. I varied both fill height h and release height H. The optimal fill height of the bottle is around 0.06 and 0.07 meters, which is roughly a third of the full height of the water bottle. This is applicable for all release heights and showed the highest success rates overall. The optimal release height from the ground or the surface on which the bottle lands was around 0.3 meters. This is roughly the smallest release height possible given the height of the bottle and the size of my wrist. I also varied the type of solution in the bottle (here ice), which I will touch upon in Chapter 3 of Discussion and Outlook. It should be noted that the standard deviation in this experiment ranged between 22 and 50%, which most likely explains the fact that I found one ideal fill height at 0.3 meters and another at 0.4 meters. To make a more precise statement I would have to throw the bottle many thousand times instead of only throwing it 30 times for each category.

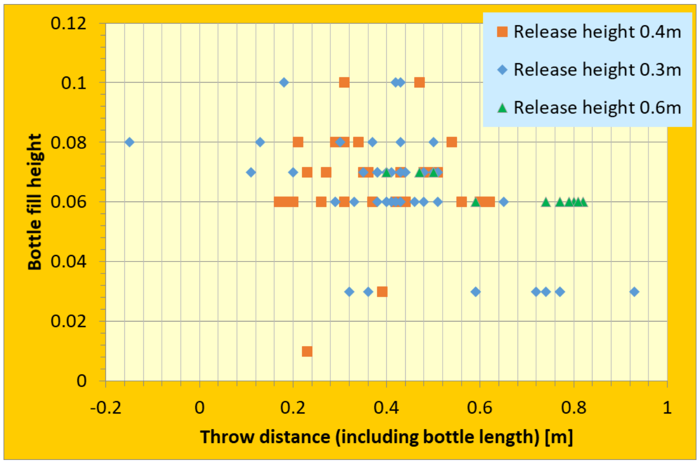


Figure 7 reports on successful water bottle flips and their throwing distance, i.e., how far it landed from the original position of the release. There seems to be a relationship between the throw distance and the fill height. I observed that the lower the fill height and the higher the release height, the higher the success rate for bottles that land further away. In contrast, the higher the fill height, the closer the bottle lands to the original throwing position. This is in line with equation 13 and 12. Bottles with a higher fill height do the opposite.

Figure 7

2.3 Comparison of Experiment and Theory

There is an excellent correspondence between the measured data and the theoretical prediction, as evidenced in by the graphs presented in this section. Here we show the results for the successful bottle flip, experiment 367, with release height 30cm and fill height 6cm. My theory reproduces the complex x-y motion (figure 8a) very well. When presented or graphed in the center of mass reference system it proves the rotation around the center of mass (figure 8b).

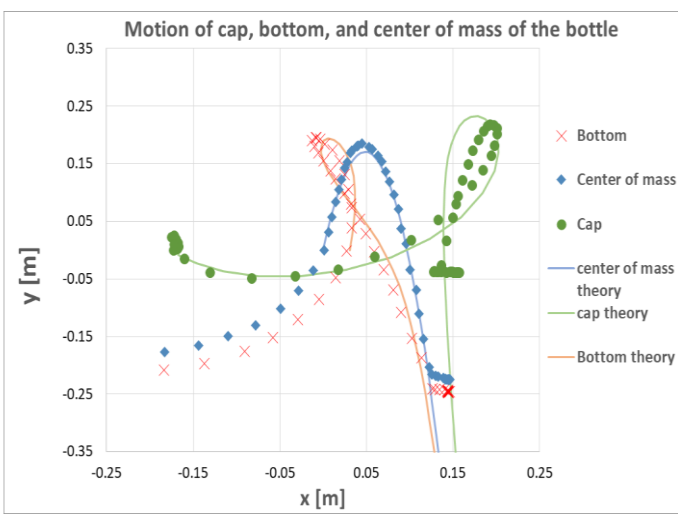


Figure 8a

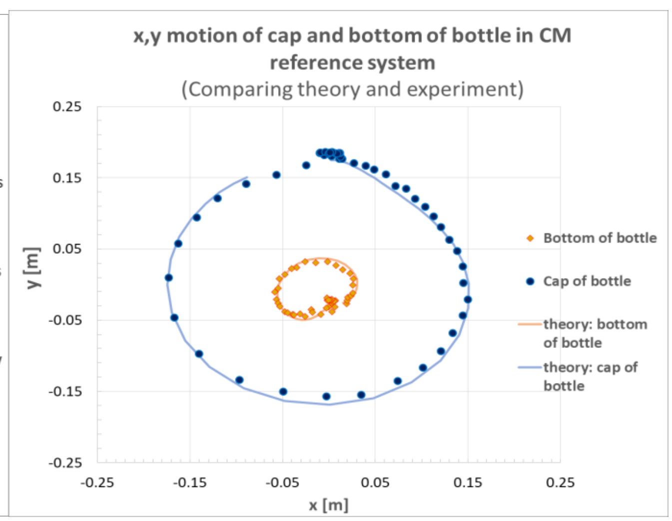


Figure 8b

Figure 9a shows the excellent correspondence between theory and experiment for the motion over time and figure 9b for the velocities. The dotted lines show the experimental data, the lines the theory. They are reasonably within error intervals ($\pm$2mm; $\pm 17ms$).

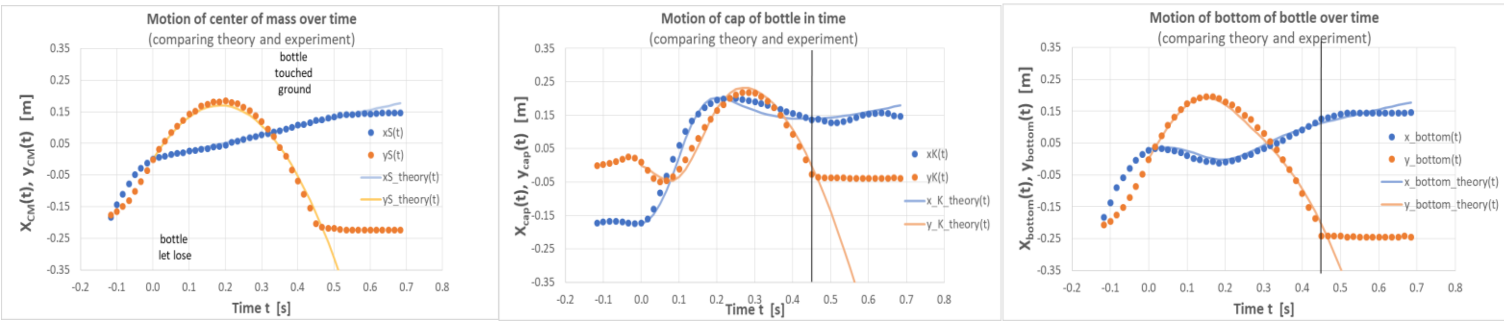


Figure 9a

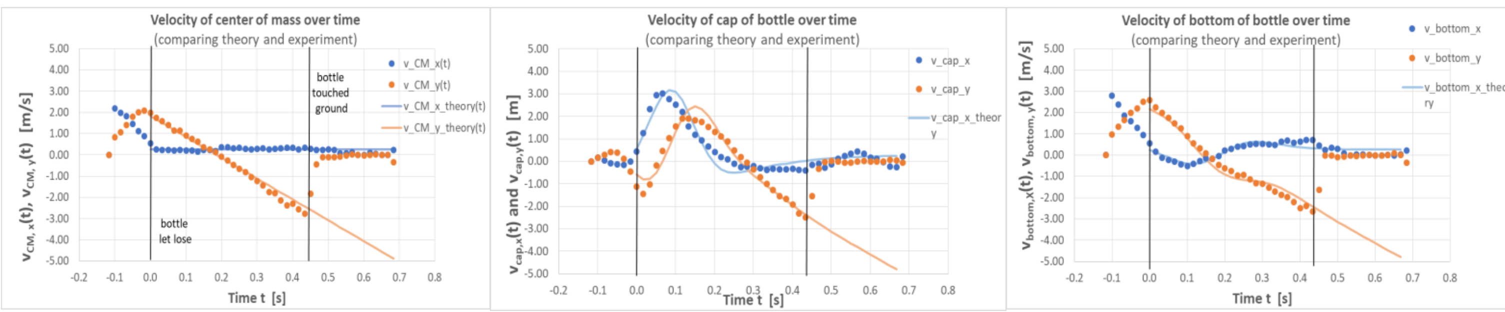


Figure 9b

Figure 10a shows the excellent fit between theory and experiment for the rotation angle φ over time. However, its first derivative – the angular velocity – is, of course, highest when the bottle is released at t=0, which, however, is not reproduced by theory which shows a maximum slightly later. This indicates a need for further investigation.

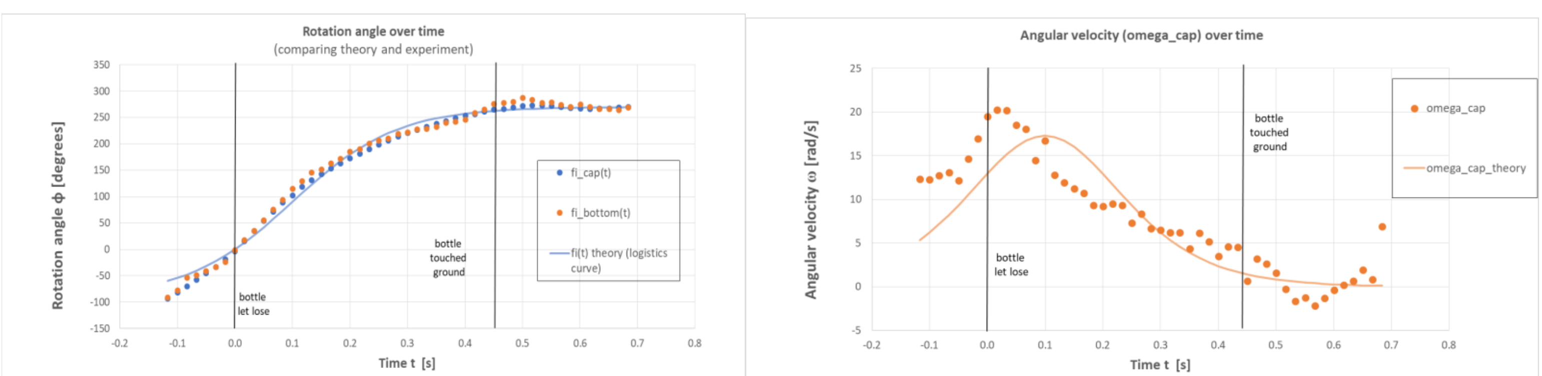


Figure 10

Figure 11

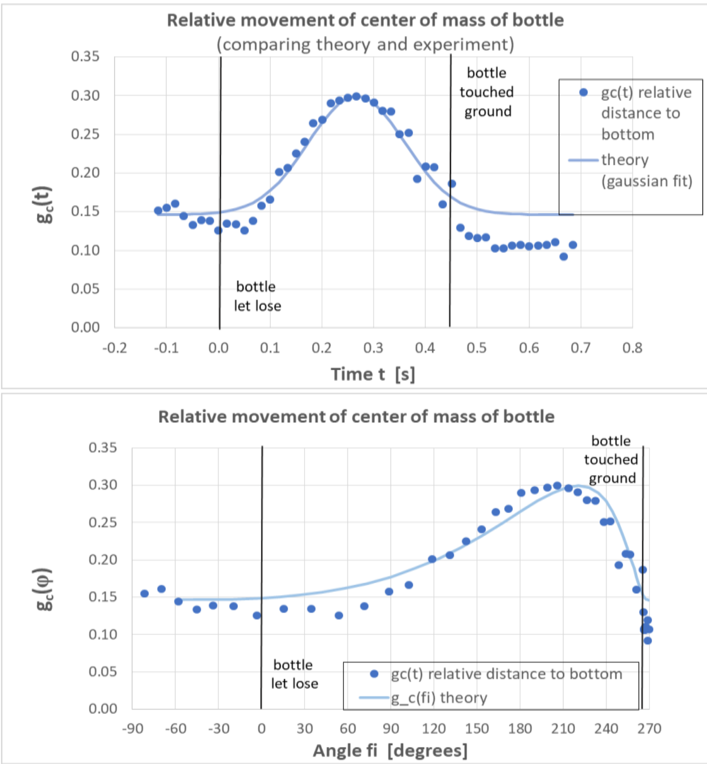


Figure 11 shows the reasonably well correspondence between theory and experiment for the relative movement of center of mass along the bottle, especially during the free throw. There may be some measurement errors before release and after landing, which need further investigation.

Finally, initial evidence shows a good experimental fit for the success conditions using the 477 recorded throws (equations 12, 13, 14, 15).

3. DISCUSSION AND OUTLOOK

I have investigated the bottle flip thoroughly. However, as in many scientific enquiries, there are still many questions which still need to be answered. In this chapter, I discuss some of them.

3.1 Lowest Center of Mass

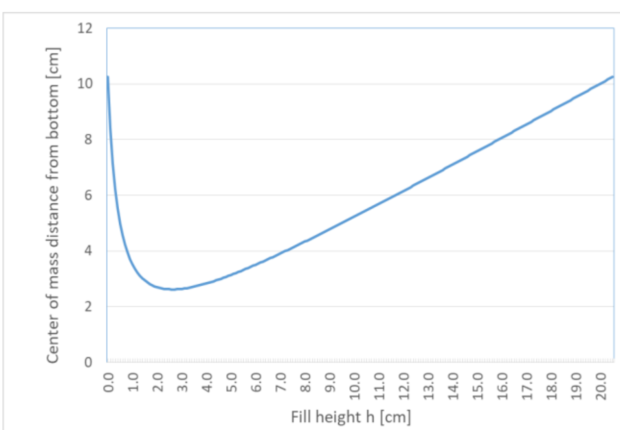


Figure 12

One key question was the optimal fill height for a successful bottle flip. Of course a low center of mass increases the probability that a bottle will stand upright. For my "Poland Spring Bottle", I easily calculated the center of mass (in terms of distance from bottom of bottle) taking into account both the mass of the water and mass of the empty bottle, see equation 14. From the plot of the center of mass CM(h) or from simply setting its first derivative 0, I determine the lowest center of mass at 2.6 cm (from the bottom (see Figure 12 and equation 14). However, from the results of the experiment, the ideal fill height with the best overall success rate is located between 6 and 7 cm. For lower fill heights, such as 3cm, the bottle becomes too unstable. This needs further investigation.

3.2 Learning Effect

Another issue to be further investigated is a possible learning effect. Because I did not use a machine to throw the bottles with exact initial velocities and spin, I need many throws to scan the parameter space. After throwing the bottle over 500 times, I must have also developed my sense of throwing and increased my skill level. This could be a factor that influenced my success rates, as it is dependent on personal skill and cannot be reproduced exactly by anyone else. However, this learning effect appears limited, as evidenced by the fact that the success rates showed the same patterns for subsequently different release heights.

3.3 Sensitivities and experimental errors

Another issue is that as a human being I did not always manage to release the bottle in the same horizontal position, but sometimes a few hundred of seconds later. I can only determine the actual release time within ±3 hundreds of a second or so. Furthermore, the angular velocity decrease quickly after the bottle is released. This poses a problem for determining the precise initial angular velocity and y-velocity, which are very important parameters determining the ultimate success of the bottle flip. More generally, the video analysis software is rather good (error intervals ±2mm; ±17ms) for tracking the well-defined bottle cap and bottom. However, the precise tracking of center of mass is trickier and suffers from higher measurement errors (the bottle is 3D, the video only 2D).

3.4 Conclusion

In conclusion, the optimal parameters for a successful flip of the “Poland Spring” bottle (20.5cm long, 12 grams) are as follows: Fill the bottle with water to a height of 6cm. Flip the bottle from a throwing height of around 30cm above ground and release it when it is approximately horizontal with an angular velocity of 19.5 rad per second and a vertical velocity component of 1.8 meters per second.

4. PLANNED WORK UNTIL GYPT

A. Complete more bottle throws and document them to reduce standard deviation.
B. Further analytical exploration of center of mass and moment of inertia.
C. Determine optimal fill height more precisely by estimating exact fill height at which including the angular velocity reaches a minimum at touch-down time t_E.
D. More comprehensive map of successful parameter combinations (eq.12, 13, 14, 15)

5. REFERENCES

[1] Vox, *the Complex Physics of that Viral Water Bottle Trick,* explained; URL: https://www.vox.com/2016/5/26/11785562/water-bottle-flip-physics

[2] *Das große Tafelwerk*, Cornelsen (2009), ISBN: 978-3-464-57146-0

[3] *Dorn-Bader, Physik Sek II, Gymnasium Gesamtband*, Westermann-Schrödel (2000), ISBN: 978-3-507-10724-3

[4] Physics Tracker 4.11.0, Opensource Software, (physlets.org/tracker), downloaded 21 Dec. 2017.